Introduction to Matching Theory

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Outline

One-to-One Two-Sided Matching without Externalities

- A society with the set of agents $\mathcal{I}$.
- *Two sides* of the market: set of agents in $\mathcal{B}$ and set of agents in $\mathcal{S}$ where the sets $\mathcal{B}, \mathcal{S}$ form a partition of $\mathcal{I}$.
- For simplicity, assume $|\mathcal{B}| = |\mathcal{S}| = n \geq 2$.
- For each $i \in \mathcal{B}$, $P^*(i)$ denotes a strict preference ordering over the elements in set $\mathcal{S}$. Similarly for each $j \in \mathcal{S}$, $P^*(j)$ denotes a strict preference ordering over the elements in set $\mathcal{B}$.
- A preference profile is denoted by $P^* = (P^*(i))_{i \in \mathcal{I}}$.
- Let $\mathcal{P}_i^*$ be the domain of preferences for agent $i \in \mathcal{I}$ and $\mathcal{P}^* = \times_{i \in \mathcal{I}} \mathcal{P}_i^*$.
- A matching is a *bijection* $\mu : \mathcal{B} \cup \mathcal{S} \to \mathcal{B} \cup \mathcal{S}$ provided:
  - $\forall i \in \mathcal{B} \cup \mathcal{S}, \mu \circ \mu(i) = i$.
  - $\forall i \in \mathcal{B}$ and $j \in \mathcal{S}, \mu(i) \in S, \mu(j) \in \mathcal{B}$.
- Denote $A(\mathcal{B}, \mathcal{S})$ as the set of all matchings.
- The triple $(\mathcal{B}, \mathcal{S}, \mathcal{P}^*)$ is called a Matching Problem without Externalities.

- A matching $\mu \in A(\mathcal{B}, \mathcal{S})$ is *(pairwise) unstable* at a preference profile $P^* \in \mathcal{P}^*$ if there exists a pair (i, j) ($i \in \mathcal{B}$ and $j \in S$) and a matching $\mu' \in A(B, S)$ such that $\mu'(i)P^*(i)\mu(i)$ and $\mu'(j)P^*(j)\mu(j)$.
- Such a pair (i, j) is called a *blocking* pair.
- If a matching μ has no blocking pairs at a preference profile $P^* \in \mathcal{P}^*$, then it is *(pairwise) stable* at P^*.
- Denote $S(\mathcal{B}, \mathcal{S}, P^*)$ as the set of all stable matchings at $P^* \in \mathcal{P}$.
- A matching $\mu' \in A(\mathcal{B}, \mathcal{S})$ *blocks* another matching $\mu \in A(\mathcal{B}, \mathcal{S})$ at $P^* \in \mathcal{P}^*$ if there exists $B \subseteq \mathcal{B}$ and $S \subseteq \mathcal{S}$ with $|B| = |S| \neq 0$ such that $\mu'(B \cup S) = B \cup S$ and $\forall i \in B \cup S$, $\mu'(i)P^*(i)\mu(i)$.
- A matching μ is in the *core* at $P^* \in \mathcal{P}^*$ if it is not blocked by any other matching.
- The set $C(\mathcal{B}, \mathcal{S}, P^*)$ at $P^* \in \mathcal{P}^*$ denote the core of the matching problem $(\mathcal{B}, \mathcal{S}, \mathcal{P}^*)$.

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$.

Example - Stability

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
s_2	s_1	s_1	b_1	b_3	b_1
s_1	s_3	s_2	b_3	b_1	b_3
s_3	s_2	s_3	b_2	b_2	b_2

- Let $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
s_2	s_1	s_1	$\mathbf{b_1}$	b_3	b_1
$\mathbf{s_1}$	s_3	s_2	b_3	b_1	$\mathbf{b_3}$
s_3	$\mathbf{s_2}$	$\mathbf{s_3}$	b_2	$\mathbf{b_2}$	b_2

- Let $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- Consider the matching $\mu = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$.

Example - Stability

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
s_2	s_1	s_1	$\mathbf{b_1}$	b_3	b_1
$\mathbf{s_1}$	s_3	s_2	b_3	b_1	$\mathbf{b_3}$
s_3	$\mathbf{s_2}$	$\mathbf{s_3}$	b_2	$\mathbf{b_2}$	b_2

- Let $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- Consider the matching $\mu = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$.
- The matching μ is not stable at P^* as (b_1, s_2) is a blocking pair.

Example - Stability

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
s_2	s_1	s_1	$\mathbf{b_1}$	$\mathbf{b_3}$	b_1
$\mathbf{s_1}$	$\mathbf{s_3}$	$\mathbf{s_2}$	b_3	b_1	b_3
s_3	s_2	s_3	b_2	b_2	$\mathbf{b_2}$

- Let $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- Consider the matching $\mu = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$.
- The matching μ is not stable at P^* as (b_1, s_2) is a blocking pair.
- Now consider the matching $\bar{\mu} = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
s_2	s_1	s_1	$\mathbf{b_1}$	$\mathbf{b_3}$	b_1
$\mathbf{s_1}$	$\mathbf{s_3}$	$\mathbf{s_2}$	b_3	b_1	b_3
s_3	s_2	s_3	b_2	b_2	$\mathbf{b_2}$

- Let $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- Consider the matching $\mu = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$.
- The matching μ is not stable at P^* as (b_1, s_2) is a blocking pair.
- Now consider the matching $\bar{\mu} = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$
- Notice that the matching $\bar{\mu}$ is stable at P^*.

- Two versions - either agents in $\mathcal{B}$ propose and agent in $\mathcal{S}$ accept or reject their proposals or viceversa.
- $\mathcal{B}$ proposing version of (DA) Algorithm:
  - First, every $i \in \mathcal{B}$ proposes to his top ranked member of $\mathcal{S}$.
  - Then, every $j \in \mathcal{S}$ who has at least one proposal is (tentatively) matched to the top $i \in \mathcal{B}$ who proposed to j and rejects the rest.
  - Then, every i who was rejected in the last round, proposes to the next best $j \in \mathcal{S}$ who have not rejected i in earlier rounds.
  - Then, every $j \in \mathcal{S}$ who has at least one proposal is (tentatively) matched to the top $i \in \mathcal{B}$ who proposed to j including any proposers tentatively matched to j from earlier rounds, (tentatively) keeps the top i amongst these proposals and rejects the rest.
  - The process is then repeated till each $j \in \mathcal{S}$ has a proposal, at which point, the tentative proposal accepted by a $j \in \mathcal{S}$ becomes permanent.
- Each $j \in \mathcal{S}$ is allowed to keep only one proposal in every round, hence each j will not be matched to more than one i.
- The algorithm will terminate at finite time since in every round the subset of $\mathcal{S}$ to whom each i can propose does not increase and strictly decreases for atleast one $i \in \mathcal{B}$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
s_2	s_1	s_1	b_1	b_3	b_1
s_1	s_3	s_2	b_3	b_1	b_3
s_3	s_2	s_3	b_2	b_2	b_2

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$, $\mathcal{S} = \{s_1, s_2, s_3\}$ and $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.

Example - Deferred Acceptance Algorithm

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
s_2	s_1	s_1	b_1	b_3	b_1
s_1	s_3	s_2	b_3	b_1	b_3
s_3	s_2	s_3	b_2	b_2	b_2

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$, $\mathcal{S} = \{s_1, s_2, s_3\}$ and $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- We illustrate the B-proposing version of the algorithm.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
$\mathbf{s_2}$	$\mathbf{s_1}$	$\mathbf{s_1}$	b_1	b_3	b_1
s_1	s_3	s_2	$\mathbf{b_3}$	$\mathbf{b_1}$	b_3
s_3	s_2	s_3	$\mathbf{b_2}$	b_2	b_2

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$, $\mathcal{S} = \{s_1, s_2, s_3\}$ and $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- We illustrate the B-proposing version of the algorithm.
- In the first round, every $i \in \mathcal{B}$ will propose to $j \in \mathcal{S}$. So, $b_1 \to s_2$, $b_2 \to s_1$ and $b_3 \to s_1$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
$\mathbf{s_2}$	$\mathbf{s_1}$	$\mathbf{s_1}$	b_1	b_3	b_1
s_1	s_3	s_2	$\mathbf{b_3}$	$\mathbf{b_1}$	b_3
s_3	s_2	s_3	$\mathbf{b_2}$	b_2	b_2

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$, $\mathcal{S} = \{s_1, s_2, s_3\}$ and $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- We illustrate the B-proposing version of the algorithm.
- In the first round, every $i \in \mathcal{B}$ will propose to $j \in \mathcal{S}$. So, $b_1 \to s_2$, $b_2 \to s_1$ and $b_3 \to s_1$.
- Hence, s_1 has two proposals: $\{b_2, b_3\}$. Since $b_3 P^*(s_1) b_2$, s_1 rejects b_2 and keeps b_3.

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
$\mathbf{s_2}$	s_1	$\mathbf{s_1}$	b_1	b_3	b_1
s_1	$\mathbf{s_3}$	s_2	$\mathbf{b_3}$	$\mathbf{b_1}$	b_3
s_3	s_2	s_3	b_2	b_2	$\mathbf{b_2}$

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$, $\mathcal{S} = \{s_1, s_2, s_3\}$ and $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- We illustrate the B-proposing version of the algorithm.
- In the first round, every $i \in \mathcal{B}$ will propose to $j \in \mathcal{S}$. So, $b_1 \to s_2$, $b_2 \to s_1$ and $b_3 \to s_1$.
- Hence, s_1 has two proposals: $\{b_2, b_3\}$. Since $b_3 P^*(s_1) b_2$, s_1 rejects b_2 and keeps b_3.
- Now, b_2 is left to choose from s_2, s_3. Since $s_2 P^*(b_2) s_3$, b_2 now proposes to s_3.

Example - Deferred Acceptance Algorithm

$P^*(b_1)$	$P^*(b_2)$	$P^*(b_3)$	$P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$
$\mathbf{s_2}$	s_1	$\mathbf{s_1}$	b_1	b_3	b_1
s_1	$\mathbf{s_3}$	s_2	$\mathbf{b_3}$	$\mathbf{b_1}$	b_3
s_3	s_2	s_3	b_2	b_2	$\mathbf{b_2}$

- Let $\mathcal{B} = \{b_1, b_2, b_3\}$, $\mathcal{S} = \{s_1, s_2, s_3\}$ and $P^* = (P^*(b_1), P^*(b_2), P^*(b_3), P^*(s_1), P^*(s_2), P^*(s_3))$.
- We illustrate the B-proposing version of the algorithm.
- In the first round, every $i \in \mathcal{B}$ will propose to $j \in \mathcal{S}$. So, $b_1 \to s_2$, $b_2 \to s_1$ and $b_3 \to s_1$.
- Hence, s_1 has two proposals: $\{b_2, b_3\}$. Since $b_3 P^*(s_1) b_2$, s_1 rejects b_2 and keeps b_3.
- Now, b_2 is left to choose from s_2, s_3. Since $s_2 P^*(b_2) s_3$, b_2 now proposes to s_3.
- Now, every woman has exactly one proposal and the algorithm stops with the matching μ^b given by $\mu^b = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$.

Theorem 1. *At every preference profile $P^* \in \mathcal{P}^*$, the DA algorithm terminates at a stable matching for that profile.*

- The $\mathcal{B}$-proposing and $\mathcal{S}$-proposing algorithms may terminate at *different* stable matchings.
- Is one *better* than the other by some criterion?
- A matching μ is $\mathcal{B}$*-optimal* (or $\mathcal{S}$*-optimal*) stable matching at $P^* \in \mathcal{P}^*$ if μ is stable and for every other stable matching μ' we have $\mu(i)P^*(i)\mu'(i)$ or $\mu(i) = \mu'(i)$ ($\mu(j)P^*(j)\mu'(j)$ or $\mu(j) = \mu'(j)$) for all $i \in \mathcal{B}$ ($j \in \mathcal{S}$).

Theorem 2. *The $\mathcal{B}$ proposing ($\mathcal{S}$ proposing) version of the DA algorithm terminates at the unique $\mathcal{B}$-optimal ($\mathcal{S}$-optimal) stable matching.*

- A matching function is a mapping $\mu : \mathcal{P}^* \to A(\mathcal{B}, \mathcal{S})$.
- A matching function μ is *manipulable* by player $i \in I$ at $P* \in \mathcal{P}^*$ via $\bar{P}_i$ if $\mu(\bar{P}_i, P^*_{-i}) P^*_i \mu(P^*_i, P*_{-i})$.
- A matching function is *strategy-proof* for every $i \in \mathcal{B}$ ($j \in \mathcal{S}$) if it is not manipulable by any $i \in \mathcal{B}$ ($j \in \mathcal{S}$).

Theorem 3. *The $\mathcal{B}$-proposing ($\mathcal{S}$-proposing) version of the DA algorithm is strategy-proof for every $i \in \mathcal{B}$ ($j \in \mathcal{S}$).*

- A matching function is *strategy-proof* if it is not manipulable by any $i \in \mathcal{I}$.
- There doesn't exist a matching that is both stable and strategy-proof.

Many-to-One Two-Sided Matching without Externalities

- In this section, we will consider *many-to-one* matching.
- We introduce the notion of *bilateral contracts* between agents in $\mathcal{B}$ and $\mathcal{S}$.
- A *bilateral contract* x is an ordered pair $(b(x), s(x))$.
- Let X be the set of all contracts.
- For every $i \in \mathcal{B}$ ($j \in \mathcal{S}$), $X_i = \{x \in X | i = b(x)\}$ ($X_j = \{x \in X | j = s(x)\}$).
- Denote $X_{\mathcal{B}} = \bigcup_{i \in \mathcal{B}} X_i$ ($X_{\mathcal{S}} = \bigcup_{j \in \mathcal{S}} X_j$).
- Each $i \in \mathcal{B}$ can sign only one contract whereas $j \in \mathcal{S}$ can hire more than one s.
- Each $i \in \mathcal{B}$ has a preference, denoted by $P^*(i)$, over the set $X_i \cup \{\varnothing\}$ where $X_j = \{x \in X | i \in \{b(x), s(x)\}\}$, $X_{\mathcal{S}} = \bigcup_{j \in S} X_j$ and $\varnothing$ is the null contract.

- Suppose the set of offered contracts is given by $X' \subseteq X$.
- The choice set of $i \in \mathcal{B}$, $C_i(X')$, is given by

$$C_i(X') = \begin{cases} \varnothing \text{ if } \{x \in X' | i = b(x), xP^*(i)\varnothing\} = \varnothing \\ \{\max_{P_i^*}\{x \in X' | i = b(x)\} \text{ otherwise} \end{cases}$$

- The choice set of $j \in \mathcal{S}$ given by $C_j(X') \subseteq \{x \in X' | j = s(x)\}$.
- Let $C_{\mathcal{B}}(X') = \bigcup_{i \in \mathcal{B}} C_i(X')$ $(C_{\mathcal{S}}(X') = \bigcup_{j \in \mathcal{S}} C_j(X'))$.
- Then the set of contracts *rejected* by $\mathcal{B}$ $(\mathcal{S})$ in X' is given by $R_{\mathcal{B}}(X') = X' \setminus C_{\mathcal{B}}(X')$ $(R_{\mathcal{S}}(X') = X' \setminus C_{\mathcal{S}}(X'))$.

- A set of contracts $X' \subseteq X$ is a stable allocation if:
  - $C_{\mathcal{B}}(X') = C_{\mathcal{S}}(X') = X'$.
  - there exists no $j \in \mathcal{S}$ and set of contracts $X'' \neq C_j(X')$ such that $X'' = C_j(X' \cup X'') \subseteq C_{\mathcal{B}}(X' \cup X'')$.

Theorem 4. *If $(X_{\mathcal{B}}, X_{\mathcal{S}}) \subseteq X^2$ is a solution to the system of equations*

$$\begin{aligned} X_{\mathcal{B}} &= X - R_{\mathcal{S}}(X_{\mathcal{S}}) \\ X_{\mathcal{S}} &= X - R_{\mathcal{B}}(X_{\mathcal{B}}) \end{aligned} \tag{1}$$

then $X_{\mathcal{B}} \cap X_{\mathcal{S}}$ is a stable set of contracts and $X_{\mathcal{B}} \cap X_{\mathcal{S}} = C_{\mathcal{B}}(X_{\mathcal{B}}) = C_{\mathcal{S}}(X_{\mathcal{S}})$. Conversely, for any stable collection of contracts X, there exists some pair $(X_{\mathcal{B}}, X_{\mathcal{S}})$ satisfying (1) such that $X' = X_{\mathcal{B}} \cap X_{\mathcal{S}}$.

- Though silent in the statement of the theorem, Theorem 4 relies on two conditions on the set of contracts - *Substitutes* condition and *Irrelevance of Rejected Contracts* condition.
- Contracts in X are *Substitutes* for $j \in \mathcal{S}$ if for all subsets $X' \subseteq X'' \subseteq X$ we have $R_j(X') \subseteq R_j(X'')$.
- In other words, the substitutes condition requires R_j to be *monotone*.
- Contracts in X satisfy the *Irrelevance of Rejected Contracts* (IRC) for $j \in \mathcal{S}$ if $\forall X' \subseteq X, \forall z \in X \setminus X'$, $z \notin C_j(X' \cup z) \Rightarrow C_j(X') = C_{\mathcal{B}}(X' \cup z)$.

Theorem 5. *Suppose contracts satisfy the substitutes condition and IRC condition, then $S(\mathcal{B}, \mathcal{S}, P^*) \neq \varnothing$.*

- The algorithm we present iteratedly solves the system of equations in (1).

- We present the $\mathcal{B}$-proposing algorithm.
- Therefore start from by setting $(X_{\mathcal{B}}(0), X_{\mathcal{S}}(0)) = (X, \varnothing)$ (i.e, players in $\mathcal{B}$ propose X and players in $\mathcal{S}$ propose nothing).
- At each stage, players in $\mathcal{B}$ and $\mathcal{S}$ holds all the acceptable offers that have been made and rejects the rest.
- We check whether $(X_{\mathcal{B}}(0), X_{\mathcal{S}}(0))$ solves the following system of equations:

$$\begin{aligned} X_{\mathcal{B}}(0) &= X - R_{\mathcal{S}}(X_{\mathcal{S}}(0)) \\ X_{\mathcal{S}}(0) &= X - R_{\mathcal{B}}(X_{\mathcal{B}}(0)) \end{aligned} \tag{2}$$

- If not, we move to the next stage by setting $(X_{\mathcal{B}}(1), X_{\mathcal{S}}(1))$ as follows:

$$\begin{aligned} X_{\mathcal{B}}(1) &= X - R_{\mathcal{S}}(X_{\mathcal{S}}(0)) \\ X_{\mathcal{S}}(1) &= X - R_{\mathcal{B}}(X_{\mathcal{B}}(0)) \end{aligned} \tag{3}$$

- We repeat this procedure, till a *fixed point* is reached.
- If the fixed point is reached in stage t, then by Theorem 4 we have a stable set of contracts given by $X_{\mathcal{B}}(t) \cap X_{\mathcal{S}}(t)$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(s_1)$	$P^*(s_2)$
s_1	s_1	$\{b_1\}$	$\{b_1, b_2\}$
s_2	s_2	$\{b_2\}$	$\{b_1\}$
		$\varnothing$	$\{b_2\}$
			$\varnothing$

- Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{S} = \{s_1, s_2\}$.

$P^*(b_1)$	$P^*(b_2)$	$P^*(s_1)$	$P^*(s_2)$
s_1	s_1	$\{b_1\}$	$\{b_1, b_2\}$
s_2	s_2	$\{b_2\}$	$\{b_1\}$
		$\varnothing$	$\{b_2\}$
			$\varnothing$

- Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{S} = \{s_1, s_2\}$.
- Let $P^* = (P^*(b_1), P^*(b_2), P^*(s_1), P^*(s_2))$.

Example - Generalized DA Algorithm

t	$X_{\mathcal{B}}(t)$	$R_{\mathcal{B}}(X_{\mathcal{B}}(t))$	$X_{\mathcal{S}}(t)$	$R_{\mathcal{S}}(X_{\mathcal{S}}(t))$
0	X	$\{(b_1,s_2),(b_2,s_2)\}$	$\varnothing$	$\varnothing$
1	X	$\{(b_1,s_2),(b_2,s_2)\}$	$\{(b_1,s_1),(b_2,s_1)\}$	$\{(b_2,s_1)\}$
2	$\{(b_1,s_1),(b_1,s_2),(b_2,s_2)\}$	$\{(b_1,s_2)\}$	$\{(b_1,s_1),(b_2,s_1),(b_2,s_2)\}$	$\{(b_2,s_1)\}$
3	$\{(b_1,s_1),(b_1,s_2),(b_2,s_2)\}$	$\{(b_1,s_2)\}$	$\{(b_1,s_1),(b_2,s_1),(b_2,s_2)\}$	$\{(b_2,s_1)\}$

- Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{S} = \{s_1, s_2\}$.
- Let $P^* = (P^*(b_1), P^*(b_2), P^*(s_1), P^*(s_2))$.
- The algorithm is initialized with $X_{\mathcal{B}}(0) = X$ and $X_{\mathcal{S}}(0) = \varnothing$.
- For $t = 1$, we start with $X_{\mathcal{B}}(1) = X$ (complement of $R_{\mathcal{S}}(X_{\mathcal{S}}(0))$) and $X_{\mathcal{S}}(1) = \{(b_1,s_1),(b_2,s_1)\}$ (complement of $R_{\mathcal{B}}(X_{\mathcal{B}}(0))$). Thus $R_{\mathcal{B}}(X_{\mathcal{B}}(1)) = \{(b_1,s_2),(b_2,s_2)\}$ and $R_{\mathcal{S}}(X_{\mathcal{S}}(1)) = \{(b_2,s_1)\}$.
- For $t = 2$, we compute $X_{\mathcal{B}}(2) = X - R_{\mathcal{S}}(X_{\mathcal{S}}(1)) = \{(b_1,s_1),(b_1,s_2),(b_2,s_2)\}$ and $X_{\mathcal{S}}(2) = X - R_{\mathcal{B}}(X_{\mathcal{B}}(1)) = \{(b_1,s_1),(b_2,s_1),(b_2,s_2)\}$. Thus $R_{\mathcal{B}}(X_{\mathcal{B}}(2)) = \{(b_1,s_2)\}$ and $R_{\mathcal{S}}(X_{\mathcal{S}}(2)) = \{(b_2,s_1)\}$.
- Repeating this procedure, we observe that $X_{\mathcal{B}}(3) = X_{\mathcal{B}}(2)$ and the process has reached a fixed point.
- Thus the algorithm terminates at round 3 and we obtain a stable set of contracts given by $X_{\mathcal{B}}(3) \cap X_{\mathcal{S}}(3)$.

Properties of GDAA: Side Optimality

- Denote the fixed points obtained from $\mathcal{B}$-proposing algorithm ($\mathcal{S}$-proposing algorithm) as $(\bar{X}_{\mathcal{B}}, \bar{X}_{\mathcal{S}})((\underline{X}_{\mathcal{B}}, \underline{X}_{\mathcal{S}}))$.
- The following theorem says that the side-optimality property that we observed in the case of one-to-one matching holds in the case of many-to-one matching as well.

Theorem 6. *Suppose contracts are substitutes for $j \in \mathcal{S}$. Then the stable set of contracts $\bar{X}_{\mathcal{B}} \cap \bar{X}_{\mathcal{S}}$ ($\underline{X}_{\mathcal{B}} \cap \underline{X}_{\mathcal{S}}$) is the unanimously most preferred stable set for every $i \in \mathcal{B}$ ($j \in \mathcal{S}$) and the least preferred stable set for every $j \in \mathcal{S}$ ($i \in \mathcal{B}$).*

- The preferences of $j \in \mathcal{S}$ satisfy the *law of aggregate demand* (LAD) if for all $X' \subseteq X''$, $|C_j(X')| \leq |C_j(X'')|$.

Theorem 7. *If the preferences of $j \in \mathcal{S}$ satisfy the substitutes condition then they satisfy the law of aggregate demand.*

- The following *rural hospital's* (RH) theorem also holds.

Theorem 8. *If the preferences of $j \in \mathcal{S}$ satisfy the substitutes condition and the law of aggregate demand then for every stable allocation $(X_{\mathcal{B}}, X_{\mathcal{S}})$ and every $i \in \mathcal{B}$ and $j \in \mathcal{S}$, $|C_{\mathcal{B}}(X_{\mathcal{B}})| = |C_{\mathcal{B}}(\bar{X}_{\mathcal{B}})|$ and $|C_{\mathcal{S}}(X_{\mathcal{S}})| = |C_{\mathcal{S}}(\bar{X}_{\mathcal{S}})|$. Here $(\bar{X}_{\mathcal{B}}, \bar{X}_{\mathcal{S}})$ refers to the fixed point obtained from the $\mathcal{B}$-proposing algorithm.*

- If the preferences of $j \in \mathcal{S}$ doesn't satisfy the law of aggregate demand then the above theorem doesn't hold.

One-to-One Two-Sided Matching with Externalities

- We incorporate *externalities* into this matching framework.
- Each agent $i \in \mathcal{I}$ has a strict preference ordering $P(i)$ over the set $A(\mathcal{B}, \mathcal{S})$.
- The set of matchings involving $i \in \mathcal{B}$ and $j \in \mathcal{S}$ is given by $A(i,j) = \{\mu \in A(\mathcal{B}, \mathcal{S}) | (i,j) \in \mu\}$.
- Let $\mathcal{P}_i$ denote the domain of preferences for player i and $\mathcal{P} = \times_{i \in \mathcal{I}} \mathcal{P}_i$.
- The triplet $(\mathcal{B}, \mathcal{S}, \mathcal{P})$ is called Matching Problem with Externalities.
- Stability of matchings in this setting crucially depends on how agents perceive others to react to their deviation.
- This idea is captured by the notion of estimation function of agents.
- Formally, an *estimation function* of agent $i \in \mathcal{B}$ is defined as a function $\varphi_i : \mathcal{S} \to 2^{A(i,j)}$.
- The set of *estimations* is given by $\varphi = \{\varphi_i | i \in \mathcal{I}\}$.

- Given φ, a matching μ is *φ-admissible* if for any pair $(i, j) \in \mu$, $\mu \in \varphi_i(j) \cap \varphi_j(i)$.
- Given φ, a matching μ is blocked by a pair $(i, j) \notin \mu$ at $P \in \mathcal{P}$ if for all $\mu' \in \varphi_i(j)$ and for all $\mu'' \in \varphi_j(i)$, $\mu' P(i) \mu$ and $\mu'' P(j) \mu$.
- A matching μ is φ-stable at $P \in \mathcal{P}$ if it is φ-admissible and has no blocking pair at P.
- The set $S_\varphi(\mathcal{B}, \mathcal{S}, P)$ at $P \in \mathcal{P}$ denotes the set of all φ-stable matchings.

- In general, we cannot guarantee the existence of φ-stable matchings.

Theorem 9. *For any $n \geq 3$, if either $\varphi_i(j) \neq A(i,j)$ or $\varphi_j(i) \neq A(i,j)$ for some $i \in \mathcal{B}$ and for some $j \in \mathcal{S}$, then there exists a preference profile $P \in \mathcal{P}$ such that $S_\varphi(\mathcal{B}, \mathcal{S}, P)) = \varnothing$.*

- The set of estimations φ is *universal* if $\forall i \in \mathcal{B}$, $\varphi_i(j) = A(i,j)$ and $\forall j \in \mathcal{S}$, $\varphi_j(i) = A(i,j)$.

Theorem 10. *If the estimations φ is universal then for every $P \in \mathcal{P}$, $S_\varphi(\mathcal{B}, \mathcal{S}, P) \neq \varnothing$.*

Example - Non-existence of φ-Stability

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$.

Example - Non-existence of φ-Stability

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$.
- Let $P = (P(b_1), P(b_2), P(b_3), P(s_1), P(s_2), P(s_3))$.

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$.
- Let $P = (P(b_1), P(b_2), P(b_3), P(s_1), P(s_2), P(s_3))$.
- Suppose $\varphi_{b_1}(s_2) = \{\mu_3\}$.

Example - Non-existence of φ-Stability

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$.
- Let $P = (P(b_1), P(b_2), P(b_3), P(s_1), P(s_2), P(s_3))$.
- Suppose $\varphi_{b_1}(s_2) = \{\mu_3\}$.
- Observe that μ_2, μ_3, μ_5 and μ_6 are blocked by (b_3, s_3).

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$.
- Let $P = (P(b_1), P(b_2), P(b_3), P(s_1), P(s_2), P(s_3))$.
- Suppose $\varphi_{b_1}(s_2) = \{\mu_3\}$.
- Observe that μ_2, μ_3, μ_5 and μ_6 are blocked by (b_3, s_3).
- Next μ_1 is blocked by (b_1, s_2).

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$.
- Let $P = (P(b_1), P(b_2), P(b_3), P(s_1), P(s_2), P(s_3))$.
- Suppose $\varphi_{b_1}(s_2) = \{\mu_3\}$.
- Observe that μ_2, μ_3, μ_5 and μ_6 are blocked by (b_3, s_3).
- Next μ_1 is blocked by (b_1, s_2).
- Lastly, μ_4 is blocked by (b_1, s_1).

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_5	μ_4	μ_1	μ_4	μ_1
μ_2	μ_6	μ_2	μ_4	μ_2	μ_6
μ_1	μ_4	μ_6	μ_6	μ_5	μ_2
μ_5	μ_1	μ_3	μ_5	μ_6	μ_5
μ_4	μ_3	μ_5	μ_3	μ_1	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$.
- Let $P = (P(b_1), P(b_2), P(b_3), P(s_1), P(s_2), P(s_3))$.
- Suppose $\varphi_{b_1}(s_2) = \{\mu_3\}$.
- Observe that μ_2, μ_3, μ_5 and μ_6 are blocked by (b_3, s_3).
- Next μ_1 is blocked by (b_1, s_2).
- Lastly, μ_4 is blocked by (b_1, s_1).
- Hence at P, $S(\mathcal{B}, \mathcal{S}, \mathcal{P}) = \varnothing$.

No Matched Couple Veto Property (NMCVP)

- Note that in Theorem 9 the estimation functions are assumed to be exogenously given - they don't depend on preferences.
- We propose a minimal condition on the estimation function which we call *No Matched Couple Veto Matching* (NMCVP).
- An estimation function φ satisfies *No Matched Couple Veto Matching* (NMCVP) if the following conditions are satisfied: Let $(i, j), (i', j') \in \mu$ for some $\mu \in A(\mathcal{B}, \mathcal{S})$.
- If for all $k \in \mathcal{I} \setminus \{i, i', j, j'\}$ and all $\mu^k \in A(i, j) \setminus A(k, \mu(k))$, $\mu P(k) \mu^k$ then $\mu \in \varphi_i(j) \cap \varphi_j(i)$.
- The estimation function in Theorem 9 doesn't satisfy NMCVP (see the example in the previous slide).
- However, NMCVP is not a sufficient condition for the existence of stable matchings.

- A matching $\mu \in A(\mathcal{B}, \mathcal{S})$ is *Pareto optimal* at $P \in \mathcal{P}$ if there is no $\mu' \in A(\mathcal{B}, \mathcal{S})$ such that $\mu' P(i) \mu$ for all $i \in \mathcal{B} \cup \mathcal{S}$.
- The set $PO(\mathcal{B}, \mathcal{S}, P)$ denotes the set of all Pareto optimal matchings at $P \in \mathcal{P}$.
- A stable matching is not always Pareto optimal.

Theorem 11. *Consider a matching problem $(\mathcal{B}, \mathcal{S}, \mathcal{P}))$ with universal estimations φ. For any $\mu \in S_\varphi(\mathcal{B}, \mathcal{S}, P)$, if μ is Pareto dominated by another matching μ' at $P \in \mathcal{P}$ then $\mu' \in S_\varphi(\mathcal{B}, \mathcal{S}, P)$.*

- Thus, starting from any stable matching we can reach a stable and Pareto optimal matching within finite steps.

Theorem 12. *For any matching problem $(\mathcal{B}, \mathcal{S}, \mathcal{P})$ with universal estimations, then at any $P \in \mathcal{P}$, $S_\varphi(\mathcal{B}, \mathcal{S}, P) \cap PO(\mathcal{B}, \mathcal{S}, P) \neq \varnothing$.*

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$.
- Suppose all agents have the same preference:
  $\mu_2 P \mu_3 P \mu_1 P \mu_4 P \mu_5 P \mu_6$.
- Then $S(\mathcal{B}, \mathcal{S}, P) = \{\mu_1, \mu_2, \mu_3\}$ but only μ_2 is Pareto optimal at P and others are not.

Core and φ-Stability

- As in the case of matching without externalities, core and φ-stability are not equivalent in the presence of externalities.
- A coalition is a pair (B, S) of non-empty subsets of $\mathcal{B}$ and $\mathcal{S}$ respectively such that $|B| = |S|$.
- A matching μ is blocked by a coalition (B, S) at $P \in \mathcal{P}$ if there exists $\mu' \in A(B, S)$ such that for any $\mu'' \in A(B^c, S^c)$ with $\mu' \cup \mu'' \neq \mu$, $\mu' \cup \mu'' P(i) \mu \; \forall i \in B \cup S$.
- The core, $C(\mathcal{B}, \mathcal{S}, P)$, is the set of all matchings that are not blocked at $P \in \mathcal{P}$ by any coalition.
- Clearly at any $P \in \mathcal{P}$, $C(\mathcal{B}, \mathcal{S}, P) \subseteq S_\varphi(\mathcal{B}, \mathcal{S}, P)$.
- In general, we cannot guarantee the non-emptiness of $C(\mathcal{B}, \mathcal{S}, P)$.

Example - Non-existence of Core

$P(b_1)$	$P(b_2)$	$P(b_3)$	$P(s_1)$	$P(s_2)$	$P(s_3)$
μ_6	μ_2	μ_6	μ_1	μ_5	μ_5
μ_4	μ_5	μ_4	μ_4	μ_6	μ_6
μ_2	μ_6	μ_2	μ_2	μ_4	μ_4
μ_1	μ_4	μ_1	μ_5	μ_2	μ_2
μ_5	μ_1	μ_5	μ_6	μ_1	μ_1
μ_3	μ_3	μ_3	μ_3	μ_3	μ_3

- Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$.
- $A(\mathcal{B}, \mathcal{S}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where
  $\mu_1 = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$, $\mu_2 = \{(b_1, s_1), (b_2, s_3), (b_3, s_2)\}$,
  $\mu_3 = \{(b_1, s_2), (b_2, s_1), (b_3, s_3)\}$, $\mu_4 = \{(b_1, s_2), (b_2, s_3), (b_3, s_1)\}$,
  $\mu_5 = \{(b_1, s_3), (b_2, s_1), (b_3, s_2)\}$, $\mu_6 = \{(b_1, s_3), (b_2, s_2), (b_3, s_1)\}$.
- Let $P = (P(b_1), P(b_2), P(b_3), P(s_1), P(s_2), P(s_3))$.
- Observe that at P, μ_3 is blocked by the grand coalition, μ_1 is blocked by $\{b_2, b_3, s_2, s_3\}$, μ_2 is blocked by $\{b_1, b_3, s_1, s_2\}$, μ_4 is blocked by $\{b_1, b_2, s_2, s_3\}$, μ_5 is blocked by $\{b_1, b_2, s_1, s_3\}$ and μ_6 is blocked by $\{b_2, b_3, s_1, s_2\}$.
- Hence at P, $C(\mathcal{B}, \mathcal{S}, P) = \varnothing$.

Thank You